Quadratic forms, Jordan algebras and the Wishart distribution

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Abstract

Celebrated Lukacs (1955) characterization of the gamma distribution was extended to Wishart distribution in Olkin and Rubin (1962), and more recently in Casalis and Letac (1996), Bobecka and Wesołowski (2002) and Boutoria, Hassairi, Massam (2010). In all these results different versions of original condition of independence of $X+Y$ and $X/(X+Y)$ for independent random variables X and Y were considered, as for example independence of $X+Y$ and $(X+Y)^{-1/2}X(X+Y)^{-1/2}$ for independent positive definite random matrices.

It seems that the very essence of the Wishart distribution is hidden not in independence properties mentioned above, but rather in an invariance property of regression of quadratic forms in the following sense: Let the space $\mathcal{Q}$ of quadratic forms on $\mathbb{R}^n$ be splitted in a direct sum $\mathcal{Q}_1 \oplus \ldots \oplus \mathcal{Q}_k$, le X and Y be independent random vectors in $\mathbb{R}^n$, let there exist a real number a such that $\mathbb{E}(X|X+Y) = a(X+Y)$ and real distinct numbers $b_1, ..., b_k$ such that $\mathbb{E}(q(X)|X+Y) = b_i q(X+Y)$ for any q in $\mathcal{Q}_i$. Then $\mathcal{Q} = \mathcal{Q}_1 \oplus \mathcal{Q}_2$, $\mathbb{R}^n$ can be structured in a Euclidean Jordan algebra and X and Y have Wishart distributions on symmetric cones related to the Jordan structures. Moreover, the subspaces $\mathcal{Q}_1$ and $\mathcal{Q}_2$ can be characterized as eigenspaces of an intriguing linear operator Ψ acting on the space of symmetric endomorphisms of the Euclidean Jordan algebra. Theses results will appear soon in Letac and Wesołowski (2010).

Keywords

Backward elimination, Censored regression, Local mis-specification, Model screening.

References

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